APPLICATION OF THE MATRIX EXPONENTIAL KERNEL

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Abstract

A point matrix kernel for radiation transport, developed by the transmission matrix method, has been used to develop buildup factors and energy spectra through slab layers of different materials for a point isotropic source. Combinations of lead-water slabs were chosen for examples because of the extreme differences in shielding properties of these two materials.

Introduction

The transmission matrix method as applied to radiation transport theory was developed by Yarmush et.al. (ref. 1). This development was applicable for uncharged particles from an infinite plane source in slab geometry. Angular distributions were approximated by angular cosine polynomial expansions and energy distributions were approximated by a group structure.

Description of the Point Kernel

As a result of the development (ref. 1), the transmission through a plane slab is

$$\underline{\phi}_{p1} = T(x)\underline{S} \qquad (1)$$

where:

$T(x)$ = the transmission matrix
$\underline{S}$ = a source vector representing the source angular and energy distribution
$\underline{\phi}$ = a response vector representing the transmitted angular and energy distribution.

Recent work (ref. 2) has shown that one can obtain a point kernel from equation (1). This kernel is

$$K_{pt} = \frac{1}{4\pi x^2}\left[T(x) - xT'(x)\right]. \qquad (2)$$

After application of the transmission matrix for shields consisting of two different materials, one has (ref. 1)

$$\underline{\phi} = T_2(x_2)[I - R_1(x_1)R_2(x_2)]^{-1}T_1(x_1)\underline{S} \qquad (3)$$

where:

$R_i(x_i)$ = reflection matrix for material i
$x_i = a_i x$
a_i = the fraction of total thickness in material i.

By definition material one is located next to the source. In a manner similar to analyses which lead to equation (2), one can write, for the transmission matrix through two slabs,

$$4\pi x^2\,\underline{\phi}_{pt} = \left(I - x\frac{d}{dx}I\right)\left\{T_2(x_2)[I - R_1(x_1)R_2(x_2)]^{-1} \times T_1(x_1)\right\}\underline{S}. \qquad (4)$$

Performing the differentiation results in

$$\begin{aligned}4\pi x^2\,\underline{\phi}_{pt} = \Big[&T_2(x_2)\,M(x_1, x_2)T_1(x_1)\\ &- x_2T_2'(x_2)M(x_1, x_2)T_1(x_1)\\ &- x_1T_2(x_2)M(x_1, x_2)T_1'(x_1)\\ &- T_2(x_2)M(x_1, x_2)[x_1R_1'(x_1)R_2(x_2)\\ &+ x_2R_1(x_1)R_2'(x_2)]M(x_1, x_2)T_1(x_1)\Big]\,\underline{S}. \qquad (5)\end{aligned}$$

where

$M(x_1,x_2) = [I - R_1(x_1)R_2(x_2)]^{-1}$.

At first indication this expression appears awkward for practical application; however, since the response effects from a double reflection for gamma rays can usually be neglected, one can write

$$\begin{aligned}4\pi x^2\,\underline{\phi}_{pt} = \Big[&T_2(x_2)T_1(x_1) - x_2T_2'(x_2)T_1(x_1)\\ &- x_1T_2(x_2)T_1'(x_1)\Big]\,\underline{S}. \qquad (6)\end{aligned}$$

Equation (5) can readily be modified to include any number of slabs of materials. Neglecting the reflection effects, one has

$$4\pi x^2\,\underline{\phi}_{pt} = \left(I - x\frac{d}{dx}I\right)[T_n(x_n)\cdots T_2(x_2)T_1(x_1)]\,\underline{S}. \qquad (7)$$

Note that these expressions are matrix expressions, and the commutation property with respect to matrix multiplication does not hold in general.

The transmission matrix is given as (ref. 1)

$$T(x) = 4C_+^{-1} e^{-\Lambda x} D(x) \tag{8}$$

where

$$D(x) = (B_+ + B_- e^{-\Lambda x} C_- C_+^{-1} e^{-\Lambda x})^{-1}$$
$$= (I - B_+^{-1} B_- e^{-\Lambda x} C_- C_+^{-1} e^{-\Lambda x} + \cdots) B_+^{-1}.$$

The matrices B_+, B_-, C_+, and C_- are functions of the material scattering and absorption properties only and not of material thickness. These operators are directly related to the radiation transport equation. Therefore the thickness occurs only in the exponential matrix which is a diagonal matrix. In most practical problems the double exponential and higher products can be neglected and equation (8) becomes the asymptotic form

$$T_\infty(x) = 4C_+^{-1} e^{-\Lambda x} B_+^{-1} \tag{9}$$

and the first derivative of T(x) becomes

$$T'_\infty(x) = -4C_+^{-1} \Lambda e^{-\Lambda x} B_+^{-1}. \tag{10}$$

Therefore equation (2) becomes

$$K_{pt} = \frac{1}{\pi x^2} C_+^{-1}(I + \Lambda x)e^{-\Lambda x}B_+^{-1}. \tag{11}$$

By similar analyses equation (6) becomes

$$\pi x^2 \Phi_{pt} = \Big(4C_{+2}^{-1} e^{-\Lambda_2 x_2} B_{+2}^{-1} C_{+1}^{-1} e^{-\Lambda_1 x_1} B_{+1}^{-1}$$
$$+ 4x_2 C_{+2}^{-1} \Lambda_2 e^{-\Lambda_2 x_2} B_{+2}^{-1} C_{+1}^{-1} e^{-\Lambda_1 x_1} B_{+1}^{-1}$$
$$+ 4x_1 C_{+2}^{-1} e^{-\Lambda_2 x_2} B_{+2}^{-1} C_{+1}^{-1} \Lambda_1 e^{-\Lambda_1 x_1} B_{+1}^{-1}\Big) \underline{S}. \tag{12}$$

Similar results can be obtained for n slabs.

Results for Multi-Layered Slabs

Results for layered combinations of lead and water are illustrated in figures 1 to 8. The choice of materials was not made for any particular shield configuration but rather to illustrate a combination of shielding material extremes. Therefore they should represent a good test for a shielding calculation. In all cases a source of 1 MeV is used.

Figure 1 shows the energy fluence build-up factor for combinations of lead slabs followed by water slabs. Figures 2 and 3 are the differential energy spectra for total shield thicknesses of five and ten mean free path lengths.

Figure 4 shows the energy fluence build-up for combinations of water slabs followed by lead slabs. As the water slabs become thicker one can note the sharp drop in the build-up factor due to even very thin slabs of lead. The shape of these curves are similar to those of Kalos (ref. 3). This effect is further illustrated in the energy spectra in figures 5 and 6. These results show the effects of the photoelectric cross section of lead. In figure 6 one can note the "filtering" effect due to the K-edge of the photoelectric cross section for lead. The effect is observable only for very thin sheets of lead.

Figure 7 illustrates the energy fluence build-up factors for a combination of three slabs of lead and water. These results show the effect of moving a lead slab through a water shield. Figure 8 shows the energy spectra for a two mean free path lead slab. This figure illustrates how effectively the lead slab removes the low energy photon build-up in the preceding water slab.

Conclusion

The point matrix kernel has been shown to be a relatively simple and effective means for calculating energy spectra and build-up factors for combinations of materials in slab geometry for a point isotropic source. This particular kernel should become an effective tool for parametric and optimization studies.

In this work only gamma ray shielding is considered; however, there is no fundamental reason why the procedure will not also work for any particles whose behavior can be described by a linear transport equation. The only distinction would be in the calculation of the particle cross sections.

ACKNOWLEDGEMENT

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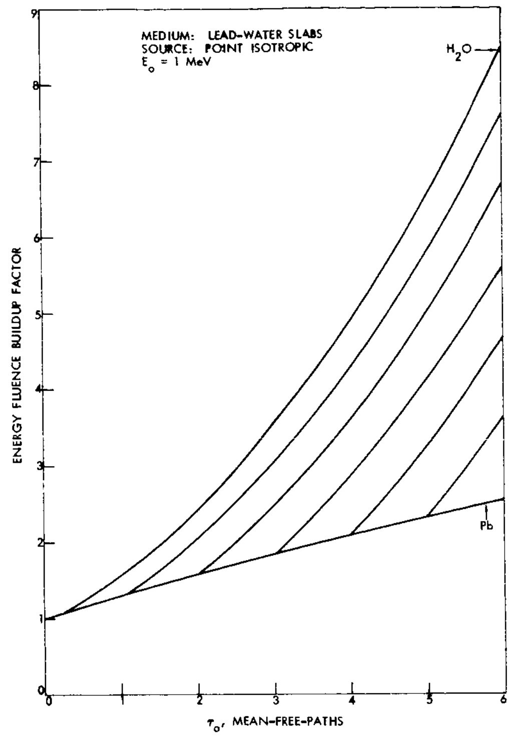
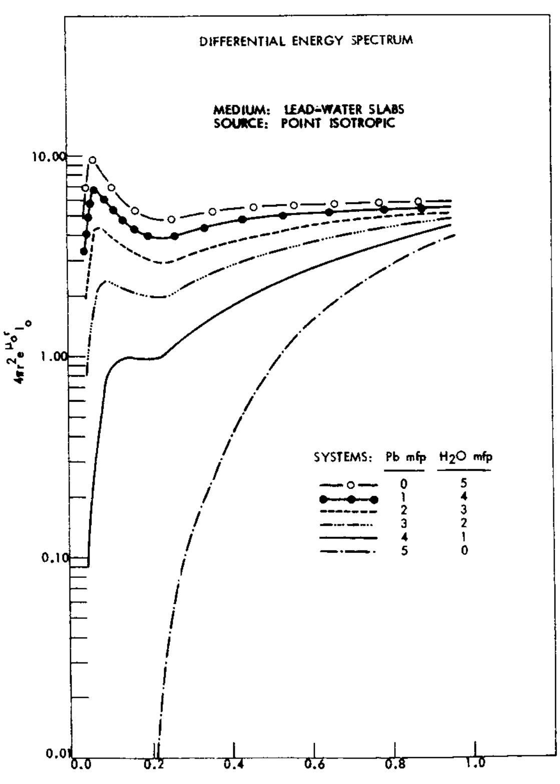
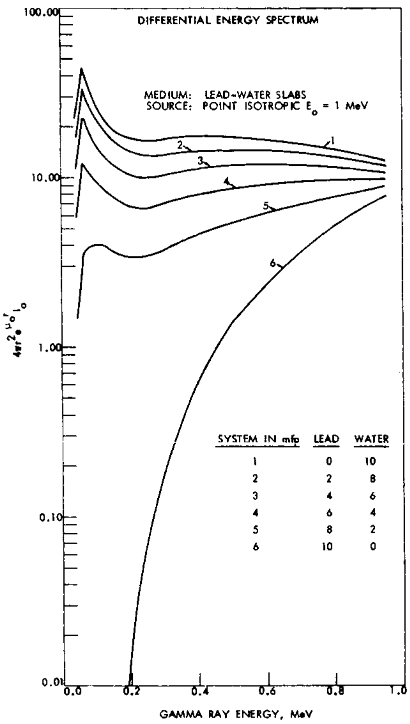
2. Rohach, A., "Analysis of the Matrix Point Kernel," submitted to Nuclear Science and Engineering, 1971.
3. Kalos, M., "Gamma Ray Penetration in Composite Slabs," Nuclear Development Corporation of American Report NDA-2056-10, 1957.



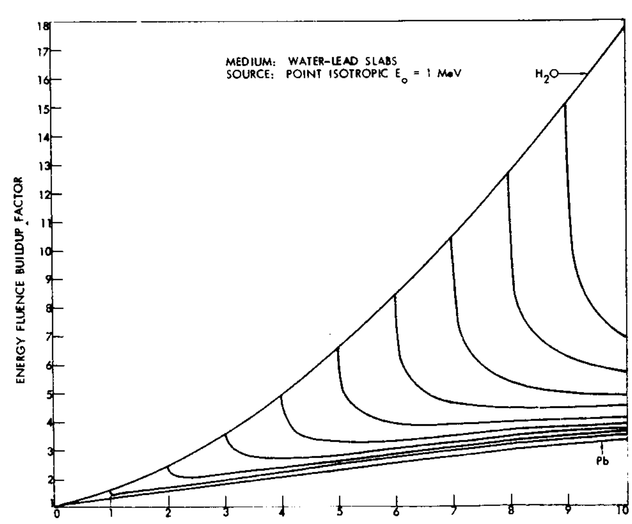
1. Energy fluence build-up factors for lead-water slabs.



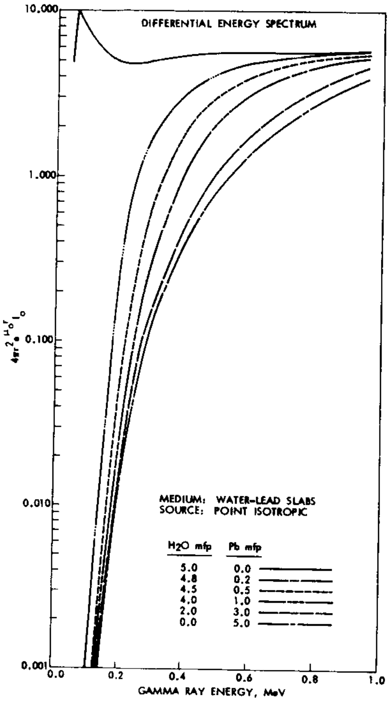
2. Energy spectra for lead-water slabs of five mean free paths total thickness.



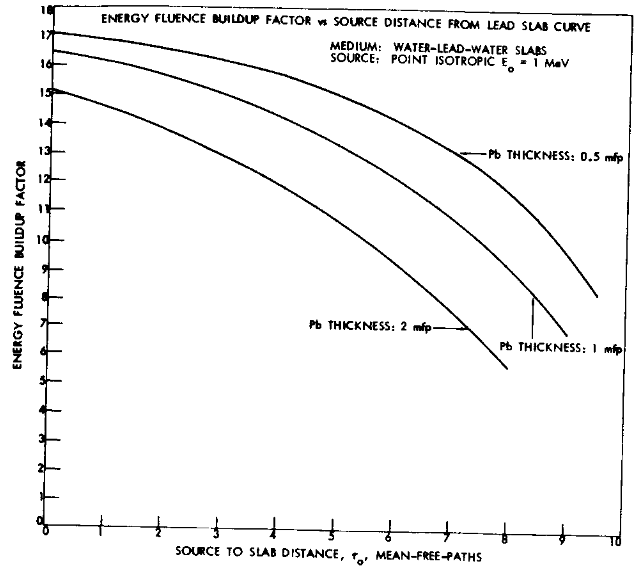
3. Energy spectra for lead-water slabs of ten mean free paths total thickness.



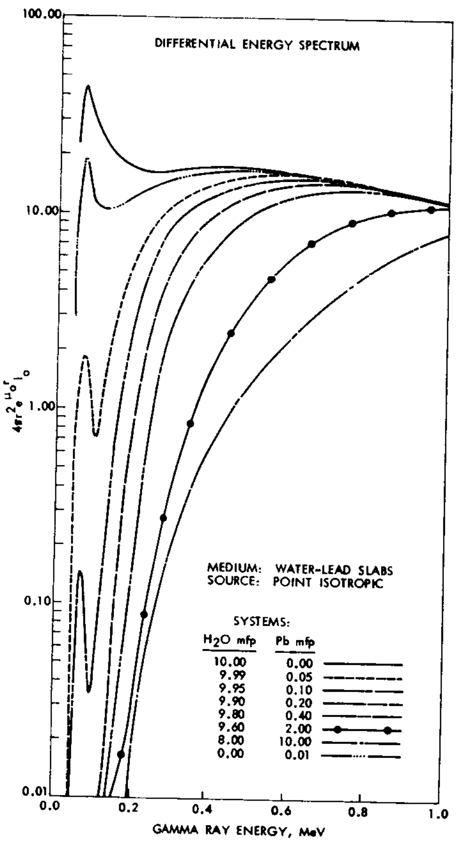
4. Energy fluence build-up factors for water-lead



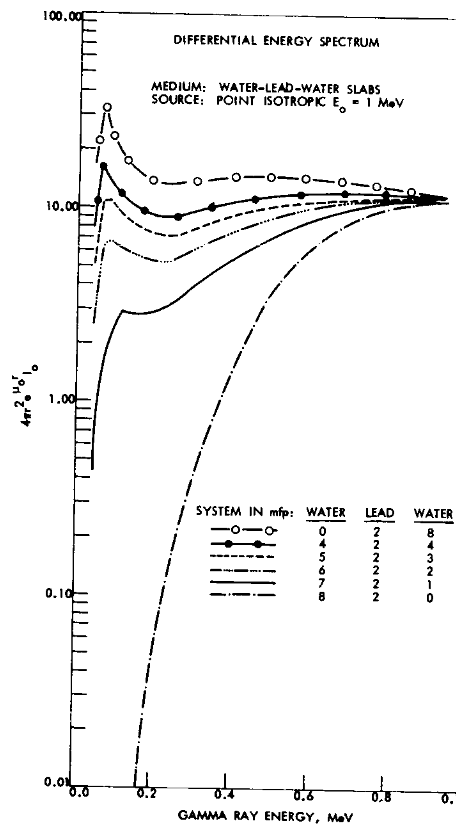
5. Energy spectra for water-lead slabs of five mean free paths total thickness.



7. Energy fluence build-up factors for water-lead-water slabs.



6. Energy spectra for water-lead slabs of ten mean free paths total thickness.



8. Energy spectra for water-lead-water slabs of ten mean free paths total thickness for a constant thickness of two mean free paths of lead.